Problem 1.2

Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

If so, *prove* it; if not, provide a counterexample (the simpler the better).

Solution

The cross product is not associative. For example, take

$$\begin{aligned}
\mathbf{A} &= \langle 0, 1, 1 \rangle \\
\mathbf{B} &= \langle 0, 1, 0 \rangle \\
\mathbf{C} &= \langle 0, 0, 1 \rangle.
\end{aligned}$$

Then

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \left(\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right) \times \langle 0, 0, 1 \rangle = \langle -1, 0, 0 \rangle \times \langle 0, 0, 1 \rangle = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 0, 1, 0 \rangle$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \langle 0, 1, 1 \rangle \times \left(\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \right) = \langle 0, 1, 1 \rangle \times \langle 1, 0, 0 \rangle = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 1, -1 \rangle.$$

Therefore,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$